

# SCALE COVARIANT GRAVITY AND EQUILIBRIUM COSMOLOGIES

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**ABSTRACT.** Causal structure, inertial path structure and compatibility with quantum mechanics demand no full Lorentz metric, but only an integrable Weyl geometry for space time (Ehlers/Pirani/Schild 1972, Audretsch e.a. 1984). A proposal of (Tann 1998, Drechsler/Tann 1999) for a minimal coupling of the Hilbert-Einstein action to a scale covariant scalar vacuum field  $\phi$  (weight  $-1$ ) plus (among others) a Klein-Gordon action term opens the access to a scale covariant formulation of gravity. The ensuing scale covariant K-G equation specifies a natural scale gauge (*vacuum gauge*). Adding other natural assumptions for gauge conditions (in particular Newton gauge, with unchanging Newton constant) the chosen Ansatz leads to a class of Weyl geometric Robertson-Walker solutions of the Einstein equation, satisfying  $a''a + a'^2 = \text{const}$ , analogous to the Friedmann-Lemaître equation but with completely different dynamical properties ( $a$  the warp function in Riemann gauge). The class has an asymptotically attracting 1-parameter subfamily of extremely simple space-time geometries with an isotropic Robertson-Walker fluid as source of the Einstein equation, discussed as *Weyl universes* elsewhere (Scholz 2005). Under the assumption of a heuristic gravitational self energy binding Ansatz for the fluid, equilibrium solutions arise, in stark contrast to classical (semi-Riemannian) cosmology. Weyl universes agree very well with a variety of empirical data from observational cosmology, in particular supernovae luminosities and quasar data.

## 1. INTRODUCTION

Equilibrium models play no role in present day cosmology. There are strong reasons for this state of affairs in the classical framework of semi-Riemannian geometry, most importantly the instability of classical static Robertson-Walker models, global singularity theorems, and the interpretation of cosmological redshift as a result of space expansion. But these are *no sufficient reasons* for an exclusive consideration of expanding space models. Already a tiny widening of the geometrical framework by integrable Weyl geometry gives a completely different picture for mathematical existence and physical acceptability of equilibrium cosmologies. They even stand in surprisingly good accord with a broad variety of empirical evidence of recent observational cosmology. Moreover, this widening of the geometrical framework corresponds to a field theoretically natural extension of classical relativity and cannot be pushed aside as a game of mathematical interest only. The following article explains why.

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Although a general acquaintance with basic ideas of localized scale gauge is presupposed here, basic notations and terminology of Weyl geometry, which are referred to in the sequel, are introduced in section 2. For the introduction of a scale gauge extension of classical relativity we follow the work of Tann and Drechsler with some adaptations (section 3). In particular the observation of the authors that in Weyl geometry a complex (or, more generally, electroweak) scalar vacuum field  $\phi$  offers excellent possibilities for forming scale invariant Lagrange densities leads to a *scale covariant version of the Einstein equation* (Tann 1998, Drechsler/Tann 1999). It transfers directly to cosmology.

Variation of the scale invariant Lagrangian density with respect to  $\phi$  (or its complex conjugate) gives a Klein-Gordon equation (section 4). Different to Drechsler/Tann we start from a Klein-Gordon action without external mass, usually regarded as a “massless” K-G field. Nevertheless there arise terms, which look like mass terms but vary with space curvature. Conformal field theory knows this very well and fixes specific choices of coupling coefficients to “get rid” of these terms. In Weyl geometry this is not necessary, due to its additional gauge freedom of scale. Here these terms express, in a very natural way, a spontaneous acquirement of mass by vacuum bosons through coupling to gravity and a biquadratic self-interaction. This proposes a natural gauge for a Higgs mechanism coupled to gravity, which has been introduced by Drechsler and Tann in a slightly different setting (section 4). Moreover, the spontaneously acquired mass term of the vacuum field allows to introduce a scale invariant mass parameter for any particle or quantum by comparison with the vacuum boson mass. It leads to a physically very plausible gauge (*vacuum gauge*) and gives reason to reconsider, at least in the cosmological context, Weyl’s old gauge hypothesis (section 5).

Next we consider Robertson-Walker geometries in this framework (section 6). The gauge conditions from section 4 and a condition satisfied by the main example of condition of section 7 specify a class of Weyl geometric Robertson-Walker geometries, which fulfill  $a''a + a'^2 = \text{const}$  ( $a$  the warp function in Riemann gauge). This class has dynamical properties which are completely different from Friedmann-Lemaitre models. In particular, there exists an asymptotically attracting 1-parameter subfamily of intriguingly simple space-time geometries, discussed as *Weyl universes* elsewhere, e.g., (Scholz 2005). The latter are derived from the well known “static” Robertson-Walker manifolds by superimposing a constant scale connection which expresses cosmological redshift (*Hubble connection*).

We now turn to the r.h.s. of the Einstein equation and study properties of highly symmetric scale covariant isentropic fluids. It is possible to find a mathematically consistent expression for negative self binding energy, dependent on mass density (“case 2”, section 7). That leads to a link between the vacuum energy tensor and mass energy density, which may be of wider import than just this Ansatz (section 8). It ensures a *stable equilibrium solution* for the corresponding Robertson-Walker fluid solution of the Einstein equation (section 9).

Such equilibrium solutions are of cosmological interest only, if they are considered in the Weyl geometric framework. Then they go together with

cosmological redshift. The material interpretation of the case 2 fluid of section 7 leads to a satisfying implementation of the Mach principle. The inertial and metrical structure of Minkowski space arises here by abstraction from a Weyl universe with critical mass density (section 10). The article ends with short remarks about empirical aspects of *Einstein-Weyl* universes (Weyl universes of positive sectional curvature). It concludes that, in the light of the Weyl geometric perspective, equilibrium solutions of cosmology should neither be neglected for cosmological theory building, nor for the analysis and interpretation of data from observational cosmology (section 10).

## 2. GEOMETRIC PRELIMINARIES

Weyl geometry has a long tradition. It started with H. Weyl's extension of (semi-)Riemannian geometry by a localized scale gauge (Weyl 1918, Weyl 1919). By many reasons, most importantly coherence with quantum physics (Audretsch e.a. 1984), the restricted version of an integrable scale connection (*integrable Weyl geometry*, IWG) is now exclusively used for physical purposes. P.A.M. Dirac enhanced it conceptually in an early attempt to bridge the gap between nuclear forces (the strong force) and gravity (Dirac 1973). Once in a while and in varying contexts, IWG has been taken up by scientists pursuing different research programs in mathematical physics or geometry. To list just a few:<sup>1</sup> (Ehlers/Pirani/Schild 1972) based their foundational and conceptual clarification of general relativity upon it. (Canuto e.a. 1977) and (Bouvier/Maeder 1977) followed the line of field theoretic and cosmological investigations opened up by Dirac. This program produced interesting partial results and has found active protagonists until the present (Tiwari 1989, Israelit 1999). (Tiwari 2003) tries to understand electron matter by Weyl geometry. (Santamoto 1984, Santamoto 1985), on the other hand, started a new approach which links the length scale with Bohm-type guiding fields and quantum potentials. That idea has been taken up by (Castro 1992, Castro/Mahecha 2006) and was extended to questions of geometric quantization. In (Castro 2006) even a mixed Brans-Dicke and Weyl geometric approach has been used in search for understanding "dark energy". (Tann 1998) and (Drechsler/Tann 1999, Drechsler 1999) have proposed a semiclassical field theoretic approach to understand mass generation by the breaking of Weyl symmetry. Finally (Folland 1970, Varadarajan 2003) and others discuss Weyl geometry from the point of view of differential geometry. Studies in conformal geometry like (Frauendiener 2000) are helpful for technical questions of rescaling.

Already this sporadic list shows that the conceptual extension of Riemannian geometry, made possible by Weyl's local scaling idea, has no canonical and unique physical application. Its intriguing mathematical and conceptual design can be made fruitful for a variety of purposes, with different chances for success. Here our main goal is to gain a broader, and perhaps

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<sup>1</sup>There are many more works on Weyl geometry in mathematical physics and geometry; the list raises no claim for representativity or even completeness. By well known reasons, even Weyl's own early intended application of WG as a unifying framework of gravity and electromagnetism has been omitted from this short list.

deeper, understanding of cosmological redshift and its relation with the vacuum structure, similar to a proposal made public by P. Cartier in his talk (Cartier 2001). From the variety of works quoted, Tann's and Drechsler's contain ideas and results which link most closely to our perspective.

A general acquaintance with Weyl geometry will be presupposed.<sup>2</sup> Here only the main terminological conventions and notations used in the sequel can be listed.

We work on differentiable manifolds  $M$ ,  $\dim M = 4$ , endowed with a Weylian metric, i.e., an equivalence class  $[(g, \varphi)]$  of pairs  $(g, \varphi)$  constituted by a semi-Riemannian metric  $g$  of Lorentz signatur  $(-, +, +, +)$  and a differential 1-form  $\varphi$ . In local coordinates the latter are given by  $g_{\mu\nu}$  and  $\varphi_\mu$ . A choice of  $(g, \varphi)$  is called a (scale) *gauge* of the metric. A change of representative from  $(g, \varphi)$  to  $(\tilde{g}, \tilde{\varphi})$  given by

$$\tilde{g} = \Omega^2 g, \quad \tilde{\varphi} = \varphi - d \log \Omega$$

is called (*scale*) *gauge transformation*, where  $\Omega > 0$  is a strictly positive real function on  $M$ .  $g$  is the *Riemannian component* of the Weyl metric and  $\varphi$  its *scale* (or *length*) *connection*. The connection is *integrable*, iff

$$d\varphi = 0.$$

By reasons indicated above, we only work with *integrable Weyl manifolds*. For simply connected  $M$  there exists a gauge in which the length connection vanishes,  $\tilde{g} = \lambda^2 g$ ,  $\tilde{\varphi} = 0$ , with  $\lambda(x) = e^{\int_{x_0}^x \varphi(c'(u)) du}$ ,  $c(u)$  any differentiable path from a fixed reference point  $x_0$  to  $x$ . This gauge is called *Riemann gauge* (in the physical literature often also called *Einstein gauge*), and  $\lambda$  the *length transfer function* of the gauge  $(g, \varphi)$ . That allows to look at the integrable Weyl manifold from the point of view of (semi-)Riemannian geometry, but it does not force us to do so.

The concepts of differential geometry, known from the (semi-) Riemannian case, can be transferred to the Weyl geometric case. The calculation of the *covariant derivative*  $D_{[(g, \varphi)]} F$  of an ordinary, i.e., scale invariant, vector or tensor field  $F$  with respect to the Weylian metric  $[(g, \varphi)]$  includes terms in  $\varphi$ , but the result is a scale invariant vector or tensor field. The same is true for Weylian *geodesics*  $\gamma_W$  and for the *curvature tensor*  $R = R_{\beta\gamma\delta}^\alpha$ , which turn out to be invariant under scale gauge transformations. But often we need concepts which allow to compare metrical measurements at different points of  $M$  in different gauges. For this purpose the length transfer function  $\lambda$  can be used.<sup>3</sup>

The Weyl structure of  $M$  allows to consider (real, complex etc.) *Weyl functions*  $f$  and (vector, tensor, spinor ...) *Weyl fields*  $F$  on  $M$ , which transform under gauge transformations by

$$f \mapsto \tilde{f} = \Omega^k f, \quad F \mapsto \tilde{F} = \Omega^l F.$$

<sup>2</sup>See, among others, (Weyl 1919, Bergmann 1942, Dirac 1973, Canuto e.a. 1977, Israelit 1999, Eisenhart 1949, Folland 1970, Varadarajan 2003, Scholz 2004, Scholz 2005). *Erratum*: In formulas (5), (6) for Ricci and scalar curvature of (Scholz 2005) the covariant derivative  $\nabla$  has to be taken with respect to the Riemannian component of the gauge  $(g, \varphi)$  only,  $\nabla = \nabla^g$ .

<sup>3</sup>More details in (Scholz 2004).

$k$  and  $l$  are the (scale or Weyl) *weights* of  $f$  respectively  $F$ .<sup>4</sup> We also write

$$[[f]] := k, \quad [[F]] := l$$

for the Weyl weight of functions or fields. With the curvature tensor  $R = R^\alpha_{\beta\gamma\delta}$  of the Weylian metric also the Ricci curvature tensor  $Ric$  is scale invariant, while scalar curvature

$$\overline{R} = g^{\alpha\beta} Ric_{\alpha\beta}$$

is of weight  $[[\overline{R}]] = -2$ .

For any nowhere vanishing Weyl function  $f$  on  $M$  with weight  $k$  there is a gauge (unique up to a constant), in which  $\tilde{f}$  is constant. It is given by

$$(1) \quad \Omega = f^{-\frac{1}{k}}$$

and will be called *f-gauge* of the Weylian metric. There are infinitely many gauges; some of them are of particular importance. An  $\overline{R}$ -gauge (in which scalar curvature is scaled to a constant) exists for manifolds with nowhere vanishing scalar curvature. It will be called *Weyl gauge*, because Weyl assigned it a particularly important role in his foundational thoughts about matter and geometry (Weyl 1923, 298f.) (cf. section 4).

Any semi-Riemannian manifold can be considered in the extended framework of (integrable) Weyl geometry. Contrary to a widespread opinion it makes sense, under certain circumstances, to do so. The rest of this article will give a first taste. For cosmological studies, Robertson-Walker manifolds are particularly important. There we have the diffeomorphism

$$M \approx R \times S_\kappa$$

with  $S_\kappa$  a Riemannian space of constant sectional curvature  $\kappa$ , here usually (but not necessarily) simply connected. If in spherical coordinates  $(r, \Theta, \Phi)$  on  $S_\kappa$

$$(2) \quad d\sigma_\kappa^2 = \frac{dr^2}{1 - \kappa r^2} + r^2(d\Theta^2 + \sin^2 \Theta d\Phi^2)$$

denotes the metric on the spacelike fibre, the Weylian metric  $[(g, \varphi)]$  on  $M$  is specified by its Riemann gauge  $(\tilde{g}, 0)$  like in standard cosmology:

$$(3) \quad \tilde{g} : \quad ds^2 = -d\tau^2 + a(\tau)^2 d\sigma_\kappa^2$$

Here  $\tau = x_0$  denotes a well chosen local or global coordinate (cosmological time parameter of the semi-Riemannian gauge) in  $\mathbb{R}$ , the first factor of  $M$ . In the semi-Riemannian perspective  $a(\tau)$ , the *warp function* of  $(M, [(g, \varphi)])$ , is usually interpreted as an expansion of space sections. The Weyl geometric perspective shows that it need not. For example, for any Weyl-Robertson-Walker manifold there is a gauge  $(g_w, \varphi_w)$  in which the “expansion is scaled away” by

$$(4) \quad g_w = \Omega_w^2 g \quad \text{with} \quad \Omega_w := \frac{1}{a}.$$

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<sup>4</sup>Mathematically, Weyl functions and Weyl fields are themselves equivalence classes of “ordinary” (scale invariant) functions and fields.

With

$$t := \int^{\tau} \frac{du}{a(u)} = h^{-1}(\tau) \quad \text{and its inverse function } h(t) = \tau$$

we get a “static” metric

$$(5) \quad g_w(x) := -dt^2 + d\sigma_{\kappa}^2 \quad \varphi_w(x) = -d \log(a \circ h) = (a' \circ h) dt = a'(\tau(t)) dt .$$

It will be called the *warp gauge* of the Robertson-Walker manifold.<sup>5</sup> Here the cosmological redshift is no longer mathematically characterized by a warp function  $a(x_0)$  but by the scale connection  $\varphi_w$ . We shall call  $\varphi = \varphi_w$  the *Hubble connection* of the Weyl manifold (in warp gauge) (Scholz 2005).

Dirac extended the calculus on integrable Weylian manifolds by defining scale covariant derivatives of Weyl functions or Weyl fields  $F$  of any weight  $[[F]]$

$$D F := D_{[(g, \varphi)]} F + [[F]] \varphi \otimes F .$$

They are Weyl tensor fields (one order higher than  $F$ ) of unchanged scale weight  $[[D F]] = [[F]]$ . For the description of relativistic trajectories Dirac introduced *scale covariant* geodesics  $\gamma_D$ . They arise from Weyl’s scale invariant geodesics  $\gamma_W$  by reparametrization, such that the weight of the tangent field  $u := \gamma'_D$  is  $[[u]] = -1$ . Then  $g(u, u)$  is scale invariant, and with it the distance measurement by Diracian geodesics, which coincides with Riemannian distance. In this sense the danger arises that, in the Diracian approach, the gauge aspect is not taken sufficiently serious from a metrical point of view.<sup>6</sup> But it offers the great advantage that Dirac’s scale covariant geodesics have the same scale weight as energy  $E$  and mass  $m$ ,  $[[E]] = [[m]] = -1$ . Therefore mass or energy factors assigned to particles or field quanta can be described more easily in a gauge independent manner in Dirac’s calculus. This was Dirac’s important contribution for facilitating the acceptance of other gauges than Riemann gauge for physical purposes, in particular for a scale covariant theory of gravity.

### 3. SCALE COVARIANT GRAVITY

A *causal structure* of space-time  $M$ ,  $\dim M = 4$ , is specified by a conformal structure  $[g]$  of a Lorentz type semi-Riemannian metric  $g$  on  $M$  ( $\text{sign}(g) = (3, 1)$ ). An *inertial structure* on  $M$  is given by a projective path structure  $\{[\gamma]\}$  of (arbitrarily parametrized) paths  $\gamma$  with any initial condition in  $TM$ , the tangent bundle of  $M$ . If the causal structure and the inertial structure satisfy certain compatibility conditions,<sup>7</sup>  $[g]$  and  $\{[\gamma]\}$  specify uniquely a *Weylian metric*  $[(g, \varphi)]$  on  $M$  (Weyl 1921, Ehlers/Pirani/Schild 1972). Compatibility of the gauge metric structure  $[(g, \varphi)]$  with quantum mechanics demands the constraint of *integrability* of the metric,  $d\varphi = 0$ ,

<sup>5</sup>In earlier publications also called “Hubble gauge”, (Scholz 2005).

<sup>6</sup>Don’t identify physical metric with the gauge invariant distances!

<sup>7</sup>Compatibility conditions are analyzed more closely in (Ehlers/Pirani/Schild 1972). Most importantly: If the initial condition of a path  $\gamma$  lies “on” the boundary  $\partial\mathcal{C}$  of a cone  $\mathcal{C}$  of  $[g]$ , the whole path  $\gamma$  is in  $\partial\mathcal{C}$ .

(Audretsch e.a. 1984).<sup>8</sup> Although the Weylian metric (integrable or not) is a fullfledged metrical structure in the mathematical sense, its scaling freedom signals an underdetermination from the physical point of view. Obviously it is necessary to specify a preferred scale gauge  $(g_o, \varphi_o)$  for observable quantities, in order to arrive at a metrically well defined physical geometry. It will be called the *observational gauge* of the theory.<sup>9</sup>

Different proposals have been presented for the choice of observational gauge in the literature. Usually they result in Riemann gauge.<sup>10</sup> But Dirac's scale covariant geodesics, combined with scale covariant derivation, show that this is not imperative (see end of last section). Here we take up the proposal of scale fixing by a scale covariant (complex) scalar vacuum Weyl field  $\phi$  on  $M$  with weight  $[[\phi]] = -1$  (Drechsler/Tann 1999) in a slightly modified form. A mass term of  $\phi$  acquired spontaneously by its specific coupling to gravity will give a natural specification for observational gauge ("vacuum gauge", see below). The Lagrangian is chosen such that  $\phi$  couples to the Einstein-Hilbert metric in the most simple way to achieve a *scale invariant* Lagrangian density. In this way, the vacuum field is related to the metric by an adapted kind of "minimal" coupling. Both together, the metric  $[(g, \varphi)]$  and the vacuum field  $\phi$ , specify a scale covariant theory of gravity including a full metrical determinaton of the metric (i.e., specification of observational gauge).

In this sense we study the dynamics on a differentiable manifold  $M$  endowed with Weylian metric  $[(g, \varphi)]$ , governed by a Lagrangian density  $\mathcal{L}$  and action

$$(6) \quad \mathcal{S} = \int \mathcal{L} dx = \int L(g, \partial g, \varphi, \phi, \phi^*, F, \rho) d\omega_g .$$

$F = (F_{\mu\nu})$  is a scale invariant  $(2,0)$  tensor field (electromagnetic field),  $\rho$  a real field of weight  $-4$  (matter density),  $\phi$  a complex scalar field (vacuum field) and  $\phi^*$  its complex conjugate, both of weight  $[[\phi]] = -1$ . We use the notation

$$d\omega_g = \sqrt{|g|} dx, \quad |g| := |\det g| ,$$

for the volume form of the Riemannian component  $g$  of the metric. In order for the Lagrangian  $\mathcal{L}$  to be scale invariant,  $L$  has to be of Weyl weight  $[[L]] = -4$ , because  $|g|$  transforms under rescaling by weight  $+8$ .

Following (Tann 1998) and (Drechsler/Tann 1999), we work with a Lagrangian density  $\mathcal{L} = L\sqrt{|g|}$ , which contains the following scale covariant

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<sup>8</sup>Compatibility of QM with  $[(g, \varphi)]$  is specified by the condition that the WKB development of a massive Klein-Gordon field on  $(M, [(g, \varphi)])$  coincides with geodesics of the metric structure, up to first order.

<sup>9</sup>In (Scholz 2005) it has been called "matter gauge" because of its directly visible effects on material measurements.

<sup>10</sup>(Ehlers/Pirani/Schild 1972, Audretsch e.a. 1984, Drechsler/Tann 1999)

versions of well known Lagrangians:

$$\begin{aligned}
L_{HE} &= \alpha \overline{R}(\phi^* \phi) && \text{Hilbert-Einstein action} \\
L_{\Lambda} &= -2\alpha\beta(\phi^* \phi)^2 && \text{cosmological term (factor } -2\alpha \text{ for convenience)} \\
L_{KG} &= \gamma D^\mu \phi^* D_\mu \phi && \text{scalar field action without (external) mass} \\
L_{em} &= \delta F_{\mu\nu} F^{\mu\nu} && \text{electromagnetic action} \\
L_m &= L_m(\rho) && \text{matter action}
\end{aligned}$$

$\alpha, \beta, \gamma, \delta$  are scale invariant coupling constants; therefore all contributions of  $L$  are of scale weight  $-4$ .  $L_m$  is a *classical action* of a cosmic matter flow; it depends exclusively on matter density  $\rho$  (not on  $\phi$ ).

Our Lagrangian differs crucially from Weyl's original Ansatz which used a quadratic term in  $\overline{R}$  to obtain the correct weight. In our approach the Hilbert-Einstein term appears coupled to the vacuum field. That leads to scale invariance without raising the power of  $\overline{R}$ .

The total Lagrangian is:

$$\begin{aligned}
(7) \quad \mathcal{L} &= \mathcal{L}_{HE} + \mathcal{L}_{\Lambda} + \mathcal{L}_{KG} + \mathcal{L}_{em} + \mathcal{L}_m = L\sqrt{|g|}, \\
L &= \alpha \overline{R}(\phi^* \phi) - 2\alpha\beta(\phi^* \phi)^2 + \gamma D^\mu \phi^* D_\mu \phi + \delta F_{\mu\nu} F^{\mu\nu} + L_m,
\end{aligned}$$

where  $\frac{\delta \mathcal{L}_m}{\delta \phi} = 0$ . It has common features with the theories of Dirac and of Brans-Dicke, but differs from both. Dirac introduced a ‘‘Lagrangian multiplier’’  $\beta$  of scale weight  $[[\beta]] = -2$ , which formally played the role of  $\phi^* \phi$ . He used it for a peculiar search for gauges expressing his ‘‘large number hypothesis’’. Brans-Dicke’s scalar field action contains an additional factor of type  $(\phi^* \phi)^{-1}$ . Similar to Tann’s and Drechsler’s, our scalar field action is that of a Klein-Gordon field,  $\mathcal{L}_{KG}$ , but here without external mass. That is an important difference to Brans-Dicke theory. In addition, we have reasons for the estimation  $\gamma \ll ||\rho_{crit}||$  (section 4), in strong contrast to Brans-Dicke theory (where  $\gamma \sim 1$ ).

On the other hand, the Lagrangian has a close kinship to conformal Klein-Gordon field theory for the massless case,  $\mathcal{L}_m = 0$ . For  $\gamma = \frac{2(n-1)}{n-2} \alpha$ ,  $n = \dim M$ , it even is conformally invariant (Tann 1998, equ. (190)), (Carroll 2004, 394f.),  $\gamma = 3\alpha$  for  $n = 4$ . Tann and Drechsler have taken this as the starting point, for what they call ‘‘mass generation by breaking Weyl symmetry’’ (Drechsler/Tann 1999). That is an interesting point of view; but their research program makes it necessary to introduce an ad-hoc external mass term into the Lagrangian of the Klein-Gordon field (see below). Here we start more modestly, also more in line with Weyl geometry, without any prior specification of  $\gamma$ . The conformal invariance is broken already for  $\mathcal{L}_m = 0$  and the invariance group of the total Lagrangian (including mass term) remains the (localized) scale extended Lorentz group  $G = \mathbb{R}^+ \times SO(1, 3)$ , i.e., the gauge group of a principal bundle over  $M$  with fibre  $G$ . If things go well,  $\gamma$  may be specified at a later stage. In the next section we shall see that the vacuum field  $\phi$  specifies a naturally distinguished gauge. This has nothing to do with ‘‘symmetry breaking’’; it rather allows a *natural fixing of the gauge*.

Weyl geometric terms in  $\varphi$  come into the play only for the variation of  $\mathcal{L}_{HE}$  and  $\mathcal{L}_{KG}$  with respect to  $g$ . The variation for the (semi-) Riemannian



case is well known from standard literature on GRT, e.g. (Carroll 2004, Straumann 2004, Weinberg 1972, Hawking/Ellis 1973). Calculations of the Weyl geometric case in (Tann 1998, Drechsler/Tann 1999) show that even for  $L_{HE}$  and  $L_{KG}$  the results are similar in form to those in Riemannian geometry, although now they contain terms in the scale connection. After adaptation of coefficients and if necessary signs because of signature change, we arrive at:

$$\frac{\delta \mathcal{L}_{HE}}{\sqrt{|g|} \delta g^{\mu\nu}} = \alpha \phi^* \phi (Ric - \frac{\bar{R}}{2} g)_{\mu\nu} \quad (\text{Drechsler/Tann 1999, (2.16)})$$

$$\frac{\delta \mathcal{L}_\Lambda}{\sqrt{|g|} \delta g^{\mu\nu}} = \alpha \beta (\phi^* \phi)^2 g_{\mu\nu} \quad (\text{Drechsler/Tann 1999, (2.16)})$$

$$\frac{\delta \mathcal{L}_{KG}}{\sqrt{|g|} \delta g^{\mu\nu}} = \gamma \left( D_{(\mu} \phi^* D_{\nu)} \phi - D_{(\mu} D_{\nu)} (\phi^* \phi) - g_{\mu\nu} \left( D^\lambda D_\lambda (\phi^* \phi) - \frac{1}{2} D^\lambda \phi^* D_\lambda \phi \right) \right) \quad (\text{Drechsler/Tann 1999, (2.17)})$$

$$\frac{\delta \mathcal{L}_{em}}{\sqrt{|g|} \delta g^{\mu\nu}} = 2\delta \left( -F_{\mu\lambda} F_\nu^\lambda + \frac{1}{4} g_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right) \quad (\text{Drechsler/Tann 1999, (2.18)})$$

(Compare (Straumann 2004, (2.90), (2.101)), or others.) In order to connect to established knowledge on couplings, we demand that

$$(8) \quad \alpha = (8\pi g_N)^{-1} [c^4], \quad \delta = (8\pi)^{-1}$$

( $g_N$  Newton constant; in the sequel we suppress factors in the velocity of light,  $c$ ). We thus arrive at the following *gauge covariant form of the Einstein equation*

$$(9) \quad Ric - \frac{1}{2} \bar{R} g = 8\pi g_N (\phi^* \phi)^{-1} \left( T^{(m)} + T^{(em)} + T^{(KG)} \right) + T^{(\Lambda)}$$

with r.h.s terms:

$$(10) \quad T_{\mu\nu}^{(m)} = -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \quad \text{matter tensor}$$

$$(11) \quad T_{\mu\nu}^{(em)} = \frac{1}{4\pi} \left( F_{\mu\lambda} F_\nu^\lambda - \frac{1}{4} g_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right) \quad \text{e.m. energy stress}$$

$$(12) \quad T_{\mu\nu}^{(KG)} = \gamma \left( D_{(\mu} D_{\nu)} (\phi^* \phi) - D_{(\mu} \phi^* D_{\nu)} \phi + g_{\mu\nu} \left( D^\lambda D_\lambda (\phi^* \phi) - \frac{1}{2} D^\lambda \phi^* D_\lambda \phi \right) \right) \quad \text{K-G energy stress}$$

$$(13) \quad T_{\mu\nu}^{(\Lambda)} = -\beta (\phi^* \phi) g_{\mu\nu} \quad (= -\Lambda g_{\mu\nu}) \quad \Lambda\text{-tensor (cf. (15))}$$

The left hand side (l.h.s.) of equation (9) is gauge *invariant*. The building blocks of the r.h.s are gauge covariant only, but the gauge weights of the factors cancel,  $[[\phi^* \phi]^{-1}] = 2$ ,  $[[T]] = -2$ . Thus the whole r.h.s. is gauge invariant, as it must be.

$$(14) \quad G_N := g_N (\phi^* \phi)^{-1}$$

may be considered as *scaled version of Newton's gravitational constant*. It is a scalar Weyl function of weight 2 in accordance with the dimensional

weight of  $g_N$ . In *this respect* our approach takes up a common motif of Dirac and of Brans-Dicke theory (Brans/Dicke 1961, 929) (also of (Scholz 2005)). Don't forget, however, that neither the form of the scalar field action nor the order of magnitude of the coupling coefficient coincides with the Brans-Dicke approach. As a consequence, the modification of classical general relativity is here much less drastic than in the latter.

A cosmological specification of our approach in terms of Weyl universes (cf. section 6) has low velocity and weak field approximations which, expressed by parameters of parametrized post-Newtonian gravity (Will 2001), leads to  $\alpha_1, \dots, \alpha_4 \approx 0$ ,  $\zeta_1, \dots, \zeta_3 \approx 0$ ,  $\beta, \gamma \approx 1$ , up to terms at the order of cosmological magnitude ( $\sim H_0$ ) (Scholz 2005, 16f). Thus dynamical predictions agree, inside the observational error margins, with those of classical relativity and with the results of high precision observations. An exception may be seen, at first sight, in the anomalous frequency shift of the Pioneer spacecrafts. It finds an extremely easy explanation in the Weyl geometric framework (Scholz 2007). But it turns out to be of non-dynamic origin and is therefore no exception to the statement on basic dynamical agreement of Weyl universe dynamics with classical relativity on the solar system level.

With

$$(15) \quad \Lambda := \beta (\phi^* \phi)$$

we get the known form

$$T^{(\Lambda)} = -\Lambda g$$

for the r.h.s. “cosmological” term. Here  $\Lambda$  is no “true”, i.e., gauge invariant constant, but a scalar Weyl function of weight -2 (it has to be, in order to arrive at a gauge invariant term  $\Lambda g$ ). The dependence of  $\Lambda$  on the norm of the gravitational vacuum field  $\phi$  expresses an indirect relationship between the  $\Lambda$ -term and the mass-energy and field energy content of the r.h.s. of (9). Below we shall see that (in vacuum gauge)  $\phi$  varies with the scalar curvature of the metric. At least under specific conditions (case 2, section 8),  $T^{(\Lambda)}$  may be considered as an expression for self-energy and stresses of the gravitational field. It would be interesting to see, whether this interpretation can be generalized.

Up to a (true) constant there is exactly one gauge in which the norm of the vacuum field is constant. It is often helpful, although not always necessary, to normalize such that

$$(16) \quad \phi^* \phi = \text{const} = 1 .$$

By obvious reasons this choice, canonically specified by the vacuum field, will be called the *Newtonian constant gauge* or simply *Newton gauge*. In this gauge we have

$$G_N = \text{const} = g_N .$$

Contrary to a widespread belief, Newton gauge *need not* necessarily coincide with Riemann gauge of the Weylian metric.

In any application of Weyl geometry it remains the question which gauge should be considered as the one which expresses empirical measurements and their scales. At the beginning of this section it has been introduced as observational gauge. Standard gravity has considered it self-evident that

Riemann gauge *is* the observational gauge. Combining this identification with the empirically well corroborated observation that the Newton constant does not change, even over cosmological time (Will 2001) and spatial distances, a silent claim of the standard approach becomes apparent. It can be stated as the following identification:

**Silent claim of (semi-)Riemannian gravity (SCRG).** *Observational gauge coincides with Riemann gauge and Newton gauge (constant  $G_N$ ). In Weyl geometric terms*

$$(17) \quad \varphi = 0 \iff \phi^* \phi = 1$$

In fact, this is nothing but a hidden hypothesis. We shall see that there are good reasons to admit alternatives and to study them seriously.

From the semi-Riemannian case we know that in Riemann gauge

$$\text{div } g^{-1}(\text{Ric} - \frac{\overline{R}}{2}g) = 0$$

in the sense of covariant divergence *div*. Scale-covariant differentiation of scale covariant tensors leads to scale covariant tensors, and the Einstein tensor is even scale invariant. Thus the result holds for any gauge and we find infinitesimal energy momentum conservation in the sense of

$$(18) \quad \text{div } T = 0 ,$$

with  $T$  the total r.h.s energy momentum tensor of (9).

$\varphi$  is no dynamical field in the integrable Weyl geometric frame, but a non-dynamic gauge freedom for the metric only. Therefore an independent variation of  $\varphi$  is not meaningful. Its observationally distinguished value is fixed by a gauge condition imparted by the vacuum field  $\phi$  which we turn to next.

#### 4. THE VACUUM FIELD

Variation with respect to  $\phi$  or with respect to  $\phi^*$ , is much more illuminating. H. Tann and W. Drechsler have already discussed the variation of (6) with respect to  $\phi^*$ . In the absence of strong quantum fields (outside of matter concentrations) we have

$$\frac{\delta \mathcal{L}_m}{\delta \phi^*} = 0$$

and arrive at the following scale covariant equation for the vacuum field (Tann 1998, equ. (358)), (Drechsler/Tann 1999, (2.13)):<sup>11</sup>

$$(19) \quad D^\mu D_\mu \phi = \gamma^{-1} \alpha \left( \frac{\overline{R}}{2} - 2\beta(\phi^* \phi) \right) \phi$$

Of course the situation becomes more involved if we study regions inside stars or even collapsing objects. Then a coupling to quantum matter fields may become indispensable.

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<sup>11</sup>Drechsler and Tann do not consider the massless case but add an ad hoc external mass term; see below.

The r.h.s factor <sup>12</sup>

$$(20) \quad M_0^2 := \gamma^{-1} \alpha \left( \frac{\bar{R}}{2} - 2\beta(\phi^* \phi) \right) = \gamma^{-1} (8\pi g_N)^{-1} \left( \frac{\bar{R}}{2} - 2\Lambda \right)$$

is equivalent to a quadratic mass term of a Klein-Gordon equation with mass  $m_0$ , where  $M_0 = m_0 c \hbar^{-1}$  (in inverse length units). It is *acquired spontaneously* by the vacuum field, due to its coupling to gravity,  $\alpha \bar{R}(\phi^* \phi)$ , and its biquadratic self interaction  $-2\alpha\beta(\phi^* \phi)^2$ .

Its scaling behaviour is what we expect from mass,  $[[M_0]] = [[m_0]] = -1$ , respectively energy  $E$ , if we construct our theory such that the Planck constant is a true constant (which implies  $[[E]] = -[[T]]$  because of  $E = \hbar\nu$ ). Therefore in Weyl geometry there is no reason to “define it away” like in the conformal approach. The scale dependent mass terms even specify a distinguished gauge in which  $M_0$  becomes constant (see below, vacuum gauge).

Although the curvature terms in the brackets of (20) are cosmologically small,  $m_0$  may be considerable or even large, depending on the order of magnitude of the inverse coupling factor  $\gamma^{-1}$ . From observational cosmology we have learned that the term  $\mathcal{L}_{KG}$  is negligible for cosmological calculations. All we need to arrive at observationally valid cosmological models is  $\mathcal{L}_{HE} + \mathcal{L}_m + \mathcal{L}_\Lambda$ . Drechsler’s and Tann’s analysis shows that in spite of this  $\mathcal{L}_{KG}$  should not be omitted. This term seems to *play an important role for the coupling of matter and interaction fields to gravity* (perhaps only in cosmologically “small” regions?).

In a first rough orientation, we may expect the derivations of  $\phi$  at comparable orders of magnitude as  $\phi$  itself. Because of the negligibility of  $\mathcal{L}_{KG}$  for cosmology at large, we conclude from comparison of  $T^{(KG)}$  and  $T^{(\Lambda)}$

$$||\gamma|| \ll ||\rho_{crit}|| ,$$

( $||\dots||$  the numerical value stripped from physical dimensions). Including dimensions  $[\gamma] = [EL^{-1}]$ , the above estimation means

$$(21) \quad \gamma \ll ||\rho_{crit}|| [L^2] ,$$

where  $[ \dots ]$  denotes physical dimensions and  $E, L, T$  energy, length, time quantities.

In order to find a bosonic mass for the vacuum field we *need not* plug an (external) mass term into the Klein-Gordon action of the vacuum field, as assumed by Drechsler and Tann. They have argued that physical gauge is due to an external mass term added to the Klein-Gordon action, and that all other masses are determined by the mass  $m_0$  of the vacuum field, which serves as a kind of “measuring rod” for energy and masses. If we take up their basic argument, but apply it to the spontaneously acquired mass  $m_0$ , we find a *physical content for the observational gauge* and, sometimes together with it, the gauge defined above as Newton gauge.

In the Weyl geometric extension of general relativity mass  $\tilde{m}$  and energy  $\tilde{E}$  of particles or quanta have to be represented by scalar quantities of weight

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<sup>12</sup>In the presence of quantum matter there may be additional terms.

$[[\tilde{m}]] = [[\tilde{E}]] = -1$ , i.e., as Weyl functions. They stand in a fixed proportion to one another, in particular to the mass of the vacuum boson,

$$(22) \quad \frac{\tilde{m}}{m_0} = \text{const} =: m, \quad \tilde{m} = m \cdot m_0.$$

Here  $m$  is a scale invariant constant.

If the spontaneous acquirement of mass according to (20) is of physical relevance, material measurements will indicate the scale invariant proportionality factor  $m$  rather than  $\tilde{m}$ , which is its appropriate scale covariant theoretical expression. The trajectory of particles is described by Dirac's scale covariant geodesics  $\gamma_D(\tau)$  and the invariant mass factor  $m$ . The trace of scale invariant geodesics  $\gamma_W(u)$  agrees with the trace of  $\gamma_D(\tau)$ , and mass scales along  $\gamma_W(u)$  like  $\tilde{m}$ . Therefore it is a matter of taste, or convenience, which description is chosen. In this context, we generally prefer the Dirac version  $(\gamma_D(\tau), m)$ .

Of course there is a preferred gauge, in which the mass of the vacuum boson  $m_0$  is constant. By obvious reasons we call it the *constant vacuum boson energy* gauge, or simply *vacuum gauge*. It is compatible with Newton gauge in the absence of strongly varying matter fields, hence in the context of cosmic mean geometry.

The spontaneously acquired mass of the vacuum field can also be used as a natural starting point for Drechsler's derivation of a *Higgs mechanism coupled to gravity* in the Weyl geometric frame (Drechsler 1999). Drechsler proposed to characterize the vacuum structure by a 2-dimensional complex vector bundle  $E \rightarrow M$  with structure group  $SU_2 \times U(1)$ . Then the vacuum field is extended from a complex scalar (purely gravitational) one to a section of vanishing curvature in  $E$  of scale weight  $-1$ , i.e., a gravito-electroweak vacuum. Locally it appears as a "scalar" Weyl field  $\Psi$  with values in  $\mathbb{C}^2$ . By a local parallelization it can be given the form

$$\Psi^{(0)} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

with  $\Psi_1 = \text{const} = 1$  and  $\Psi_2 = \phi$ . In this way, the electroweak vacuum structure can be reduced to a complex scalar field in Weyl geometry, which satisfies a Klein-Gordon equation as above. A "reduction", more precisely a non-homomorphic mapping of the electroweak group action onto the stabilizer of  $\Psi^{(0)}$ , leads to mass terms for other electroweak boson fields with non-vanishing curvature. If this differential geometric version of the Higgs "mechanism" gives a physically correct picture of gravitational couplings of electroweak bosonic fields, we have  $m_0 = m_H$ , the Higgs mass. The experimental measurement of  $m_H$  would allow to determine the value of  $\gamma$ .

To complete the analysis of the Higgs mechanism, Drechsler used his additionally postulated (external) mass term. We have seen that Weyl geometry offers a more natural self-gauging mechanism. This could not be seen by Drechsler, because he passed over to the classical Riemann gauge shortly before the end of his analysis, like in (SCRG), and adopted the central hypothesis of traditional cosmology (17). In this way he needlessly gave away crucial advantages offered by the Weyl geometric extension of the theory.

The two artificial restrictions (last minute Riemann gauge and external boson mass) can be avoided as indicated.

## 5. VACUUM GAUGE AND WEYL'S GAUGE HYPOTHESIS

Up to now we only dealt with theoretically natural ways for gauge choices. This opened up new thought possibilities beyond the silent claim of semi-Riemannian gravity (SCRG); but it remains still completely unclear whether it leads to new physical insight. We now look for physical specification of gauge. Here our attention is mainly directed towards cosmology, but similar investigations could be brought to bear for other applications.

Drechsler's and Tann's proposal to consider the vacuum boson mass as the clue for observational gauge is very convincing. It deserves to be considered as a basic principle of scale covariant (Weyl geometric) gravity.

**Principle of vacuum gauge (VG).** *The vacuum boson mass serves as a kind of measuring rod for mass/energy in general and, with it (dualized), length, time etc. scales, i.e., observational gauge is given by vacuum gauge:*

$$(23) \quad M_0 = \text{const} \quad \text{in observational gauge}$$

High precision measurements speak strongly against a varying gravitational constant (Will 2001). Similar to the standard approach in this respect, we can therefore explore how far we come with the following, simplest possible, assumption:

**Empirical postulate (EP).** *For the empirically accessible part of the universe the value of the gravitational constant can be considered as observationally constant. We therefore assume that in 'empty' space (no strong matter fields above cosmic mean energy density) the observational gauge of a Weyl geometric approach coincides with Newton gauge,*

$$(24) \quad G_N = \text{const} \iff \phi^* \phi = 1 \quad \text{in observational gauge.}$$

Equation (20) shows that in the presence of strong matter fields (classical or quantum), which induce a varying scalar curvature  $\bar{R}$ , Newton gauge ( $\phi^* \phi = 1$ ) and vacuum gauge ( $M_0 = \text{const}$ ) *cannot* coincide. That is different in the absence of strong fields.

The principle (VG) and (EP) taken together presuppose the existence of a gauge (observational gauge) in 'empty space' for which the following holds:

$$(25) \quad M_0 = \text{const} \iff \phi^* \phi = 1 \iff G_N = \text{const}$$

Equ. (20) shows that in such a gauge also the scalar curvature  $\bar{R}$  is scaled to a constant and coincides with Weyl gauge. Therefore in any Weyl geometric approach satisfying the VG hypothesis and (EP) *Weyl gauge is of particular importance*. This gives new reasons to reconsider an old hypothesis of H. Weyl, which looked rather ad hoc in its original version and has been neglected for a long time by comprehensible reasons (Weyl 1923,

298f.).<sup>13</sup> In order to make its methodological status as clear as possible, we state it in the following form:

**Lemma 1.** *In any scale invariant approach to cosmology with Lagrangian density (7), in which (VG) and (EP) hold, Newton gauge, vacuum gauge and Weyl gauge are identical,*

$$(26) \quad \phi^* \phi = 1 \iff M_0 = \text{const} \iff \bar{R} = \text{const} ,$$

*and coincide with observational gauge.*

This is no ad hoc replacement of the the hidden hypothesis (SCRG) of the received approach. If the comparison of observable masses at different places in the world is induced by their proportions to the vacuum boson and there is no external reason (varying mass field), which changes conditions, the Weyl gauge principle (26) is due to a *self-gauging vacuum structure*. The latter serves as a kind of “measuring rod” in the sense of Drechsler/Tann’s analysis.

If this assumption is physically realistic, it will be consequential for the coupling of interaction fields with gravity also inside massive systems, in particular for collapsed objects. In Riemann gauge and in Newton gauge scalar curvature can acquire extremely high values near the critical boundary surface of a black hole (the event horizon of a future trapped achronal set in Riemann gauge), if there is inflowing mass/energy. Length and time measuring units are inversely proportional to  $M_0$  and indicate much larger volume than in Riemann gauge. We can thus expect much more volume in vacuum gauge close to the event horizon but still in the exterior. This volume expansion can be informally described as dark volume pockets (“dark”, because the geodesic structure remains unchanged, only the Riemannian component of the metric is modified). This may lead to different metrical and dynamical properties in the environment of the classical event horizon than expected in the present theory. Under our assumptions, the singular behaviour of the semi-Riemannian structure may be naturally dissolved by a more physical rescaling of the metric. In this sense, a kind of physical “resolution of singularities” may take place for the singularity neighbourhoods which characterize galactic cores and quasars in Riemannian geometry.<sup>14</sup>

## 6. COSMOLOGICAL MEAN GEOMETRY

For the description of cosmological mean geometry, i.e., under abstraction from local deviations by inhomogeneities of mass distribution, it makes sense to assume the existence of a (local or even global) fibration of space-time

$$\tau : M \longrightarrow T \approx \mathbb{R}$$

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<sup>13</sup>Much weaker, but already recognizable, in the 4th edition, translated into English (Weyl 1922, 308f.)

<sup>14</sup>Seyfert galaxies seem to indicate a transition phase of galactic cores to active mass energy ejection, when the effective energy density has sufficiently decreased by gravitational binding effects and by the conjectured metrical effects of vacuum gauge, to allow strong outflowing systems of trajectories (jets).

with spacelike fibres  $S_t := \tau^{-1}(t)$ ,  $t \in T$ , which are maximally homogeneous and isotropic, and homothetic among each other. Then the Riemann gauged manifold  $(M, (\tilde{g}, 0))$  can be isometrically characterized by a warped product with warp function  $a$

$$M \cong \mathbb{R} \otimes_a S_\kappa ,$$

where  $S_\kappa$  is a 3-dimensional space of constant sectional curvature,  $\kappa \in \mathbb{R}$ . Topologically we have  $M \approx \mathbb{R} \times S_\kappa$ , and  $\tilde{g}$  can be given by a Robertson-Walker metric like in (3).

In this case,  $T^{(em)} = T^{(KG)} = 0$ , and the r.h.s. of the Einstein equation reduces to

$$(27) \quad 8\pi G_N T := 8\pi G_N T^{(m)} + T^{(\Lambda)}$$

It is well known that here  $T$  has the form of a fluid energy momentum tensor

$$T = (\mu + p)X^* \otimes X^* + p g ,$$

with homogeneous and isotropic mass-energy density  $\mu$  and pressure  $p$ , which can be expressed in terms of the warp function  $a$  and its first two derivatives  $a', a''$  (O'Neill 1983, 346).<sup>15</sup> It can be assimilated to equ. (27) in two ways,

$$\begin{aligned} T^{(m)} &= T^{(fl)} , & T^{(\Lambda)} &= 0 , & \text{fluid of pressure } p, \text{ mass density } \mu, \\ T^{(m)} &= \rho_m X^* \otimes X^* , & T^{(\Lambda)} &= \Lambda g , & \text{dust and } \Lambda\text{-term} \\ & & & \text{with } \rho_m = \mu + p \text{ and } \Lambda = -8\pi G_N p , \end{aligned}$$

or a linear combination of both.

Because of lemma 1, we are particularly interested in Weyl gauged Robertson-Walker manifolds, respectively fluids. Under natural assumptions for the fluid, which will be analyzed in the next section (lemma 2), warp gauge will turn out as a physically interesting case. We thus have reasons to consider the class of Weyl geometric Robertson-Walker manifolds for which Weyl gauge and warp gauge coincide. This is a strong constraint. The following proposition shows that it reduces the infinite dimensional model class to a 2-parameter subclass. The exemplars of this subclass will be called *generalized Weyl universes*.

**Proposition 1.** *A Weyl geometric Robertson-Walker manifold with warp function  $a(\tau)$  (in Riemann gauge) characterizes a generalized Weyl universe  $\iff$  the warp function satisfies the differential equation*

$$(28) \quad a''a + a'^2 = \text{const} .$$

*Proof.* The scaling function from Riemann gauge to warp gauge (5) is  $\Omega_W = \frac{1}{a}$ . Scalar curvature is of weight  $-2$ . Rescaling from Riemann gauge to Weyl gauge (1) goes thus by  $\Omega_{\bar{R}} = \bar{R}^{\frac{1}{2}}$ . With the expression for the scalar curvature of Robertson-Walker manifolds (O'Neill 1983, 345)

$$(29) \quad \frac{\bar{R}}{6} = \left( \frac{a'}{a} \right)^2 + \frac{\kappa}{a^2} + \frac{a''}{a}$$

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<sup>15</sup>  $\mu = 3(8\pi G_N)^{-1}(a'^2/a^2 + \kappa/a^2)$ ,  $p = -(8\pi G_N)^{-1}(2a''/a + a'^2/a^2 + \kappa/a^2)$



the claim of the lemma is a direct consequence of

$$\Omega_W \sim \Omega_{\overline{R}} \iff \frac{1}{a^2} \sim \left(\frac{a'}{a}\right)^2 + \frac{\kappa}{a^2} + \frac{a''}{a} . \quad \square$$

For the Weyl geometric approach, the role of this differential equation is comparable to the Friedmann-Lemaître equation for classical (semi-Riemannian) cosmology. It specifies a physically reasonable 2-parameter subclass of all Weylian Robertson-Walker solutions (for a given r.h.s *const*).

The two-parameter set of solutions of the non-linear differential equation (28) is surprisingly simple. For  $H^2 := \text{const}$  it is

$$(30) \quad a(\tau) = \pm \sqrt{(H\tau)^2 + C_1\tau + C_2} ,$$

with constants  $C_1, C_2 \in \mathbb{R}$ .

Clearly the solution class is much simpler than the Friedmann-Lemaître class. It even contains a distinguished special case. For large values of  $\tau$ , the general solution rapidly approximates the special solution  $a_0$  with  $C_1 = C_2 = 0$  (uniformly on the remaining interval  $(\tau, \infty)$  and with rapid convergence  $|(a - a_0)|_{(\tau, \infty)} \rightarrow 0$  for  $x \rightarrow \infty$ )

$$(31) \quad a(\tau) \longrightarrow a_0(\tau) := H\tau .$$

In this sense, the linear warp function  $a_0(\tau) = H\tau$  characterizes an attractor solution of the warp function for the whole class of generalized Weyl universes.

In order to pass to the warp gauge, we have to reparametrize the time coordinate  $t := H^{-1} \log H\tau \iff \tau = H^{-1}e^{Ht}$ . Then the Robertson-Walker metric with linear warp function  $ds^2 = -d\tau^2 + (H\tau)^2 d\sigma_\kappa^2$  acquires a “scale expanding” form (Masreliez)

$$\tilde{g} : \quad ds^2 = e^{2Ht}(-dt + d\sigma_\kappa^2)$$

and can be rescaled by  $\Omega = e^{-Ht}$  to

$$(32) \quad g : \quad ds^2 = -dt^2 + d\sigma_\kappa^2, \quad \varphi = Hdt$$

These models have been called (special) *Weyl universes* elsewhere (Scholz 2005), and so they will be here. Basically they can be derived from the old static models of cosmology by superimposing a time-homogenous scale connection. That results in a small deformation of the metrical structure.

If the general Weyl universe with warp function (30) is treated similarly, the Hubble gauge becomes

$$\varphi = \frac{H^2 + 2C_1He^{-Ht} + (2C_1^2 - 4C_2)e^{-2Ht}}{H + 2C_1e^{-Ht} + 4C_2e^{-2Ht}}dt \approx Hdt \quad \text{for “large” } t .$$

The rapid approximation behaviour stated above can easily be read off from this explicit expression.

We resume our result as:

**Theorem 1.** *The (special) Weyl universes (32) with curvature parameter  $\kappa$  are stable solutions of (28) in the class of generalized Weyl universes. Any other member of the class is a generalized Weyl universe of the same  $\kappa$  and approximates, for increasing  $t$ , the special Weyl universe in fast approximation (in the sup norm).*

Another result is not difficult to prove. Here it will be stated without proof, because its more mathematical in nature than the rest of this contribution.

**Theorem 2.** *Up to isomorphism, (special) Weyl universes form a 1-parameter family of models with the only essential metrical parameter (module)*

$$(33) \quad \zeta := \frac{\kappa}{H^2} .$$

The Hubble constant (here  $H = H_0 c^{-1}$ ) itself is no structural parameter of the model class. It remains, of course, a clue for the adaptation of the model to observational data.

We set

$$(34) \quad \Omega := \frac{\mu}{\rho_{crit}} ,$$

with

$$\rho_{crit} = \frac{3H^2 (\phi^* \phi)}{8\pi g_N} [c^4] = \frac{3H^2}{8\pi G_N} [c^4]$$

the critical density, as usual. Then the total net energy density,  $\Omega = \Omega_m + \Omega_\Lambda$ , determines the geometrical module  $\zeta$  by the condition (Scholz 2005)

$$(35) \quad \Omega = \zeta + 1 .$$

This relation is the Weyl geometric analogue of the well known balance for the Friedmann-Lemaitre class,  $\Omega_m + \Omega_\Lambda + \Omega_\kappa = 1$ . Here the other relative densities are

$$\Omega_m = \frac{2}{3}\Omega; , \quad \Omega_\Lambda = \frac{\Omega}{3} .$$

Equ. (29) shows  $\overline{R} = 6(\zeta + 1)H^2$ . Furthermore

$$\Lambda = \beta(\phi^* \phi) = 8\pi g_N (\phi^* \phi)^{-1} \rho_\Lambda = 3H^2 \Omega_\Lambda = H^2(\zeta + 1) .$$

For Weyl universes the crucial r.h.s. factor in (20) in observational gauge (= Weyl gauge) becomes

$$(36) \quad \frac{\overline{R}}{2} - 2\beta(\phi^* \phi) = H^2(\zeta + 1) = \kappa + H^2 .$$

That is exactly the principal sectional curvature in the spacelike fibres of the Riemann gauge (in the warped product!) (O'Neill 1983, 345). The mass of the vacuum boson in a Weyl universe with module  $\zeta$  is now given (in Weyl gauge) by

$$(37) \quad m_0^2 c^4 = \left( \frac{\hbar c}{g_N} \right) \frac{c \hbar}{8\pi \gamma} H^2(\zeta + 1) ,$$

where  $\frac{\hbar c}{g_N} = E_{Pl}^2$  is the squared Planck energy.

## 7. FLUID MASS TERM

Usually a perfect fluid *Ansatz* is chosen to describe the mass term in cosmological solutions of the Einstein equation, e.g., (Hawking/Ellis 1973, 69f.). It can easily be adapted to the Weyl geometric context. Consider a matter Lagrangian density

$$(38) \quad \mathcal{L}_m = -2\rho(1 + \epsilon)\sqrt{|g|} ,$$

where  $\rho$  is a (real) scalar field which represents the energy density. At the moment we do not analyze in which gauge it becomes constant; for special cases that will become clear later (case 2). Its scale weight is  $[[\rho]] = -4$ , in agreement with dimensional conventions for mass/energy density.  $\epsilon$  is a real scalar function of  $\rho$ , characterizing the proportion of *internal energy* to  $\rho$ . Obviously its gauge weight has to be  $[[\epsilon]] = 0$ .  $\mathcal{L}_m$  is a scale invariant density.

In normal fluids  $\epsilon$  represents the elastic potential and increases monotonically with  $\rho$ . Following (Fahr/Heyl 2007) in the search for a mathematical characterization of gravitational self binding energy, we investigate whether  $\epsilon$  may be employed as an expression for the latter (at first only formally). Increasing mass energy density  $\rho$  ought to lead to higher gravitational self binding energy which has to be subtracted from the original mass energy. That reduces the internal energy of the fluid. Different to ordinary fluids, we therefore expect here

$$(39) \quad \frac{d\epsilon}{d\rho} \leq 0 \quad \text{with equality only for } \epsilon = 0 .$$

The effective (“net”) *total energy density* of the fluid is

$$(40) \quad \mu = \rho(1 + \epsilon) .$$

Let  $X = (X^\mu)$  be a timelike unit Weyl vector field of weight  $[[X]] = -1$  and

$$j := \rho X$$

the respective mass energy current. If the condition of conserved current

$$(41) \quad \text{div } j = D_\mu j^\mu = 0$$

is satisfied, the *pressure* of the fluid is

$$(42) \quad p = \rho^2 \frac{d\epsilon}{d\rho} .$$

Variation of the metric in Riemann gauge under the restriction (41) leads to the energy stress tensor of the fluid (Hawking/Ellis 1973, 70)

$$(43) \quad T_{\mu\nu}^{(fl)} = -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} = \rho_m X_\mu X_\nu + p g_{\mu\nu} ,$$

with

$$(44) \quad \rho_m := \mu + p$$

its net mass energy density. The fluid is *isentropic*, if  $p$  is function of  $\mu$  only. According to (Hawking/Ellis 1973) this is a sufficient condition for  $T^{(fl)}$  being derivable from a Lagrangian.

All expressions in the above derivation are scale covariant.  $T^{(fl)}$  is a Weyl tensor field of weight  $-2$  and therefore compatible with (10), although it has been derived from a slightly more constrained variational principle. Therefore it seems justified to transfer the fluid Ansatz to scale covariant gravity.

Two cases are of particular interest for us.

**Case 1:** *Ordinary dust.*

No pressure, no internal energy  $\epsilon = 0$ ,  $p = 0$ ,  $\mu = \rho = \rho_m$ .

Lagrangian density

$$\mathcal{L}_m = -2\rho_m \sqrt{|g|} ,$$

mass energy tensor

$$T^{(m)} = T^{(fl)} = \rho_m X^* \otimes X^*$$

( $X^*$  the  $g$  dual of  $X$ , cf. below).

**Case 2:** *Dust with self binding energy given by  $\epsilon$ , coupled to the  $\Lambda$ -term.*

In the Lagrange density (38) we assume a negative self binding energy given by  $\epsilon$ , such that the pressure (42) is gauge covariant with weight  $[[p]] = -4$ , like  $\rho$ . That implies  $\frac{d\epsilon}{d\rho} \sim \rho^{-1}$ . We therefore assume a logarithmic self binding coefficient of the form

$$(45) \quad \epsilon = -\log \left( \frac{\rho}{\rho_{crit}} \right)^\alpha = -\alpha \log \tilde{\Omega}$$

with  $\rho > 0$ , and  $\alpha > 0$  a logarithmic power characteristic for the self binding energy. (Confusion of this coefficient  $\alpha$  with the one from the Lagrangian is excluded because of the substitution (8).) Here we use the abbreviation

$$(46) \quad \tilde{\Omega} := \frac{\rho}{\rho_{crit}} .$$

$\frac{d\epsilon}{d\rho} = -\frac{\alpha}{\rho}$  is negative, as we expect for gravitational self binding energy. Under condition (45), the pressure term becomes proportional to the original energy density:

$$(47) \quad p = \rho^2 \frac{d\epsilon}{d\rho} = -\alpha \rho$$

The *net total energy* density is

$$(48) \quad \mu = \rho(1 + \epsilon) = \rho(1 - \alpha \log \tilde{\Omega}) .$$

$\rho$  will be called “*gross*” *energy density* in the sequel.

The energy stress tensor of the fluid decomposes into two terms

$$T^{(fl)} = (\mu + p)X^* \otimes X^* + p g ,$$

with  $X^* = (X_\mu)$  the dual of  $X = (X^\mu)$  with respect to the Riemannian component of any gauge and  $[[X^*]] = 1$ . The contraction is  $\mathfrak{L}(X \otimes X^*) = X^\mu X_\mu = g(X, X) = -1$ .  $T^{(fl)}$  has the form of an ordinary dust energy tensor and a cosmological term:

$$\begin{aligned} 8\pi G_N T^{(fl)} &= 8\pi G_N T^{(m)} + T^{(\Lambda)} , \\ T^{(m)} &= \rho_m X^* \otimes X^* , \quad T^{(\Lambda)} = 8\pi G_N p g = -8\pi G_N \rho \Lambda g , \end{aligned}$$

with  $p = -\alpha\rho = -\rho_\Lambda$  and

$$(49) \quad \rho_m = \mu - \alpha\rho = \rho(1 - \alpha(1 + \log \tilde{\Omega}))$$

$$(50) \quad \rho_\Lambda = \frac{\Lambda}{8\pi G_N}, \quad \Lambda = 8\pi g_N(\phi^*\phi)^{-1}\alpha\rho.$$

At first sight it may look like a formal trick that a “cosmological” term drops out of the isentropic fluid Ansatz. But *any* mathematical descriptor for gravitational self energy ought to be a kind of cosmological correction to the ordinary relativistic r.h.s. of the Einstein equation. In this sense the metaphor of cosmic matter and “ether” (in the language of Einstein, Weyl e.a. in the 1920s) as a kind of fluid with negative pressure (Levi-Civita 1926, 359, 429) may perhaps serve as a fruitful heuristics for understanding the  $\Lambda$ -term still today. We have to keep in mind, however, that the  $\Lambda$ -term plays a distinctive role in the variation.  $\mathcal{L}_\Lambda$  is varied with respect to general  $g$  and  $\mathcal{L}_m$  under the constraint (41) only.

Scale gauge weights of all terms are consistent with what we demand for the r.h.s. tensors of equation (9).

Contraction of the l.h.s of the (1,1) Einstein equation leads to the well known term

$$\bar{R} - 4\frac{\bar{R}}{2} = -\bar{R}.$$

Contraction of the (1,1) raised energy stress tensor at the r.h.s. gives, up to the constant  $8\pi G_N$ ,

$$(\mathbb{C}(\rho_m X \otimes X^*) + 4p) = -\rho_m + 4p = -\mu + 3p = \rho(3\alpha - 1 - \epsilon)$$

The l.h.s. is constant in Weyl gauge. Differentiation of the r.h.s. shows that the latter is constant, if  $\rho' = 0$  or  $\epsilon = 1 - 2\alpha$ , which leads back to  $\rho = \text{const}$ . By (47), (48) this implies constant effective dust energy density  $\rho_m$  and constant negative pressure  $p$  and  $\Lambda$  in Weyl gauge. According to (25) this means that *in observational gauge* (which is equal to Weyl gauge by lemma 1)

$$\rho_m = \text{const}, \quad p = \text{const},$$

independently of the choice of  $\alpha$ . For Robertson-Walker fluids, i.e., fluids with the same type of symmetry as Robertson-Walker manifolds, we draw the consequence:

**Lemma 2.** *Weyl gauge and warp gauge coincide for case 2 fluids.*

*Proof.* If Weyl gauge  $(g_W, \varphi_W)$  has a Riemannian component

$$g_W : \quad ds^2 = -dt^2 + a_W(t)^2 d\sigma_W^2,$$

the (rest) warp function  $a_W$  must be constant. Otherwise the conserved current condition (41) would imply a non-constant  $\rho_m$ , in contradiction to what we have just seen. Thus Weyl gauge is the same as warp gauge.  $\square$

This property of case 2 fluids has striking consequences for the physical geometry generated by them. In the last section it has been shown that it restricts the geometry to generalized Weyl universes and then, by theorem 1, to a (special) Weyl universe geometry as a stable attractor which is approximated in fast convergence.

## 8. COUPLING OF THE VACUUM FIELD TO MASS ENERGY

A most interesting property of case 2 is its inherent coupling of the vacuum tensor to mass energy. H.-J. Fahr and M. Heyl have remarked that present cosmology presupposes a cosmic vacuum which acts on matter and geometry, without itself being acted upon by the latter (Fahr/Heyl 2007). This unsatisfactory state of affairs may be improved by investigating possible coupling conditions of  $T^{(\Lambda)}$  to matter  $T^{(m)}$ . In such a case  $\Lambda$  can no longer be a true constant. While this may seem irritating, at first glance, for the received perspective, it is no problem in the Weyl geometric program. Here we have  $\Lambda = \beta(\phi^*\phi)$  by equation (15) anyhow, and the question turns into the one, whether the vacuum field  $\phi$  may couple to matter density averaged over sufficiently large cosmological regions in a physically reasonable way.

Fahr and Heyl argue that an increasing vacuum energy density should lead to an increasing flow of particles from the sea of vacuum fluctuations to the matter component of  $T$ , i.e., matter creation. The other way round, an increase of matter density should lead to a rising gravitational self binding energy. This results in a transfer of energy from (net) gravitating matter to the gravitational field, and with it to the vacuum component of  $T$ .

These qualitatively convincing arguments are difficult to quantify because of the problem of an undefined energy of the gravitational field in general relativity and the lack of tested knowledge of matter creation from the vacuum. Fahr/Heyl propose an interesting method which explores this field; but we cannot be sure that their route will lead to a valid answer. We take up the question and their general line of approach, but explore the latter in the perspective of scale covariant gravity.

In case 2 above the total (net) energy density  $\mu = \rho + \epsilon$  has been reduced in comparison with the original (gross) mass energy density  $\rho$  by  $\epsilon = -\log \tilde{\Omega}^\alpha$  because of overall gravitational binding effects. Moreover, a part of the total energy density,  $\alpha\rho$ , is transferred from the dust (mass) term  $T^{(m)}$  to  $T^{(\Lambda)}$ . Under the assumption of (45) the term  $T^{(\Lambda)}$  contains the observable energy momentum of the gravitational self binding effects. Similarly *any negative energy fluid Ansatz (38), (39) should give a natural coupling of the vacuum tensor to mass energy* comparable to (50).

This shows that it is not at all unreasonable to expect links between partial Lagrangians. The link in our case 2 example can be described by comparing it with a pure dust matter Lagrangian (case 1)

$$\mathcal{L}_m = -2\rho_m \sqrt{|g|} \quad \rho_m = \rho(1 - \alpha(1 + \log \tilde{\Omega}))$$

and a cosmological term related to it (cf. (50)) by the condition

$$(51) \quad \beta = \frac{8\pi g_N}{(\phi^*\phi)^2} \alpha \rho, \quad \text{i.e.} \quad \mathcal{L}_\Lambda = -16\pi g_N \alpha \rho \sqrt{|g|}.$$

Clearly the *self binding exponent  $\alpha$  is here the crucial parameter* for the balance between mass energy density and the vacuum tensor.

In order to acquire a better understanding of structural possibilities for cosmological models, it may be useful to investigate such coupling conditions more generally, even independent of the special background theory of

isentropic fluids, case 2. The essential point is a coupling of type (51). More generally we know from equ. (20) that

$$\Lambda = \beta(\phi^* \phi) = \frac{\overline{R}}{4} - \frac{\gamma}{2\alpha} M_0^2 .$$

In vacuum gauge it rises with  $\overline{R}$ , and thus with mass energy density. To concentrate ideas we continue here, however, by referring to the case 2 example.

## 9. EQUILIBRIUM CONDITION

For a classical Robertson-Walker perfect fluid, i.e., a solution of the semi-Riemannian Einstein equation with r.h.s tensor (43), the expansion dynamics is regulated by Raychaudhuri's differential equation for the warp (expansion) function  $a(t)$  (O'Neill 1983, 346)

$$(52) \quad \frac{a''}{a} = -\frac{4\pi g_N}{3}(\mu + 3p) .$$

This leads to the known equilibrium condition for the Einstein universe and other non-evolving ("static") Robertson-Walker solutions with constant sectional curvature of the space sections:

$$(53) \quad 3p_0 = -\mu$$

For  $p < p_0$ , the gravitational forces of mass energy win over the negative pressure, and the corresponding Robertson-Walker cosmology will collapse in the long run. For  $p > p_0$  it will expand (Ellis 1999, A44). The balance equ. (53) shall be called the *hyle condition* of a cosmic fluid. It is satisfied for Robertson-Walker fluids  $\iff a'' = 0$ ; i.e., not only in the static case (cf. footnote (15)).

Clearly this equilibrium is attained for case 2 (cf. (47), (48)), iff

$$3\alpha\rho = \mu = \rho(1 + \epsilon) = \rho(1 - \alpha \log \tilde{\Omega}) .$$

This condition establishes a peculiar relationship between the (logarithmic) binding energy exponent  $\alpha$  and the equilibrium value  $\tilde{\Omega}_0$  of (relative) gross energy density:

$$(54) \quad \alpha = (3 + \log \tilde{\Omega}_0)^{-1} \iff \tilde{\Omega}_0 = e^{\alpha^{-1} - 3}$$

In order to understand better what its "peculiarity" consists of, we consider small deviations  $h$  of gross energy density from the hyle condition  $\rho_0 = \tilde{\Omega}_0 \rho_{crit}$ ,

$$\rho = \rho_0 + h .$$

The self binding energy will then be shifted from the equilibrium value  $\epsilon(\rho_0)$  to  $\epsilon(\rho) = -\alpha \log \tilde{\Omega}$  and the pressure from  $p_0 = -\alpha\rho_0$  to

$$p(\rho) = -\alpha(\rho_0 + h) .$$

The difference of the l.h.s. of the hyle condition is

$$\Delta(3p) = -3\alpha h$$

The change of total (net) energy density close to the hyle (equilibrium) value  $\mu_0 = \rho_0(1 - \alpha\tilde{\Omega}_0)$  is

$$\Delta\mu = h \cdot \frac{d\mu}{d\rho}|_{\rho_0} + o(h^2) .$$

Because of

$$\frac{d\mu}{d\rho} = 1 - \alpha(1 + \log \tilde{\Omega})$$

we find

$$\frac{d\mu}{d\rho}|_{\rho_0} = 1 - \alpha(1 + \alpha^{-1} - 3) = 2\alpha$$

and thus

$$(55) \quad |\Delta(3p)| > \Delta\mu \quad \text{for small } h \neq 0 .$$

For  $h > 0$  the increase of negative pressure of  $T^{(\Lambda)}$  dominates over the contracting effect of increasing gross energy density (at least in the regime where the linearized estimation is applicable). The originally increased mean density  $\rho$  is forced back towards the hyle condition  $\rho_0$ . If it passes to the other side,  $h < 0$ , the effect of (55) works the other way round;  $\rho_0$  turns out to be a stable equilibrium point.

We formulate our conclusion as

**Proposition 2.** *Under the assumptions of case 2, the hyle cosmology (53) characterizes a stable equilibrium for a Robertson-Walker fluid with given self binding energy exponent  $\alpha$ , or the neutral state of an oscillatory mode.*

If oscillations are damped by intrinsic reasons the hyle point is a (local) attractor. Notice that in the classical picture it need not be static. Its generic state is a uniform “expansion” with  $a'' = 0$ , i.e., a linear warp function

$$a(\tau) = H \tau , \quad H = \text{const}$$

and characterizes a Weyl universe (section 6).

Even if we consider case 2 as a predominantly methodological tool for investigating possible effects of matter vacuum coupling, we should not consider it as a mere “toy” example. The existence of a stable attractor may be characteristic for the model space of other, more general, gravitational binding Ansatzes for  $\mathcal{L}_\Lambda$ . It seems worthwhile to investigate which other self binding assumptions lead to comparable stability effects. From the mathematical point of view, it would be illuminating to know the maximal subspace of all Robertson-Walker fluids, in which the hyle cases are stable. Theorem 1 teaches us that generalized Weyl universes belong to it.

Stable solutions have become doubtful since Eddington’s counter argument against Einstein’s first cosmological model (Eddington 1930). The rejection has become generally accepted (Einstein included), after the interpretation of cosmological redshift as an effect of space expansion became dominant. It was even further supported by the proofs of the general singularity theorems by R. Penrose and S. Hawking in the 1960s.<sup>16</sup> *They do not apply, however, in situations where a coupling of  $\mathcal{L}_m$  and  $\mathcal{L}_\Lambda$  stabilizes the structure.* Our case 2 example shows that the convergence of past directed

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<sup>16</sup>For a crystal clear overview of the state of the arts, see (Ellis 1999).



timelike unit vector fields, which is one of the necessary premisses in the singularity theorems 2 to 4 of (Hawking/Ellis 1973), can be excluded for specific, not even particularly strange coupling conditions.

#### 10. CAUSE OF COSMOLOGICAL REDSHIFT AND THE MACH PRINCIPLE

The expanding space paradigm for the explanation of cosmological redshift does not admit equilibrium cosmologies. The last passage makes sense in a Weyl geometric context only. At other occasions it has been discussed, how well integrable Weyl geometry (IWG) suits the task of analyzing different physical assumptions for the causes of cosmological redshift from a common mathematical vantage point (Scholz 2004). Different scale gauges allow to transform one and the same cosmological redshift function  $z(\tau)$  ( $\tau$  the cosmological time parameter) into different mathematically equivalent expressions. IWG is the appropriate framework for comparing the expanding space interpretation of  $z$  by the warp function of Robertson-Walker models (3),

$$z(\tau) = \frac{a(\tau)}{a(\tau_0)} - 1 ,$$

with an energy scaling effect for photons of the Weylian length (scale) connection

$$z(\tau) = e^{\int_{\tau_0}^{\tau} \varphi(c'(u)) du} ,$$

or with a combination of both ( $c(u)$  differentiable path).

If in observational gauge, the physically most relevant one, the warp function is completely gauged away, the cause of cosmological redshift can no longer be seen in a real “expansion” of space sections. It rather appears to be a purely field theoretic effect of the interaction of photons (the Maxwell field) and the gravitational vacuum field  $\phi$ . Cosmological redshift is then a “higher order” effect of gravity, in the sense of remaining unexpressed in the semi-Riemannian approximation of the theory. Its simplest mathematical expression is given by the Weylian scale connection, which appears here as the *Hubble connection*  $\varphi$ .

Lemma 2 and theorem 1 imply that the physical geometry established by case 2 Robertson-Walker fluids is very well approximated by a Weyl universe. Its crucial parameter (mathematically  $\zeta$ , physically  $\Omega = \zeta + 1$ ) is determined by the logarithmic exponential  $\alpha$  of the gravitational self binding coefficient  $\epsilon$ . We know from eqs. (48) and (54) that

$$(56) \quad \Omega = \Omega_0 = \frac{\mu_0}{\rho_{crit}} = \tilde{\Omega}_0(1 - \alpha \log \tilde{\Omega}_0) = 3\alpha e^{\alpha^{-1}-3} .$$

Therefore the equilibrium value of the geometric module  $\zeta$  for case 2 fluids is

$$(57) \quad \zeta = 3\alpha e^{\alpha^{-1}-3} - 1 .$$

$\alpha = \frac{1}{3}$  corresponds to a spatially flat Weyl universe ( $\zeta = \kappa = 0$ , Minkowski-Weyl universe),  $\frac{1}{3} < \alpha < 1$  characterizes a subset of the Lobachevsky-Weyl case ( $0 > \zeta \gtrsim -0.6$ , for which  $\kappa < 0$ ), and  $0 < \alpha < \frac{1}{3}$  Einstein-Weyl universes with positive spatial curvature,  $\zeta > 0 \Leftrightarrow \kappa > 0$ .

Equ. (57) constrains total energy density to  $\Omega \gtrsim 0.41$ . i.e.,  $\Omega_m \gtrsim 0.27$ . Therefore a massless case 2 fluid Weyl universe,  $\Omega = 0$ , does not exist, while

the space-time of special relativity appears as a Minkowski-Weyl universe with neglected Hubble connection. It corresponds to  $\Omega = 1$ , i.e., critical total energy density. In this sense, our model class of Weyl universes has an inbuilt implementation of Mach's principle. *Even a Machian "cause" for the inertial structure of Minkowski space becomes apparent:* A total net energy density  $\mu = \rho_{crit}$ , corresponding to a logarithmic self energy exponent  $\alpha = \frac{1}{3}$ , while the Hubble connection has been "forgotten", i.e., has been abstracted from.

Although our case 2 assumptions were derived from formal considerations (scale covariance conditions for  $\rho$  and  $p$  of isentropic fluids), our analysis shows a striking (theoretical) convergence of stepwise refined conditions toward the class of Weyl universes. They have apparently such extraordinary mathematical properties that structurally similar, but more general assumptions for the coupling of the vacuum term to mass may be expected to lead to the same model class. It is tempting to conjecture that a purely gravitational self binding energy of electrically neutral matter belongs to the logarithmic exponential  $\alpha = \frac{1}{3}$ , corresponding to the Minkowski-Weyl solution. Partial ionization of the diluted matter content (mainly hydrogen) may be responsible for reducing the value. Observational values of SNIa etc. indicate a good empirical behaviour for  $\alpha \approx 0.21$ .

## 11. A SHORT LOOK AT OBSERVATIONAL DATA AND CONCLUSION

*Einstein Weyl universes* with positive spacelike curvature  $\kappa > 0$  are observationally clearly distinguished. The actual data of supernovae luminosities show a perfect fit of the Hubble diagram of Weyl universes with positive sectional curvature to the supernovae observations. Weyl universes of  $\zeta \approx 2.6$  give the best fit with the most recent SNIa data (Riess e.a. 2007). The estimated confidence interval is

$$(58) \quad \zeta \in [2.19, 3.0] .$$

Supernovae data are fitted with mean square error  $\sigma_W \approx 0.21$  (at  $\zeta = 2.6$ ), which lies *below* the mean square error of the data set  $\sigma_{dat} \approx 0.24$ . The fit quality is better than for the Friedmann-Lemaitre model class which leads to a mean square model error  $\sigma_{FL} \approx 0.27$ , with  $\Delta\sigma \approx 0.03$  *above* the data error and twice as much above  $\sigma_W$ . This difference is, however, still far from significant ( $\sigma_{FL} - \sigma_W \approx 0.2\sigma_{dat}$ ). We shall have to see what happens when measurements to higher redshift values or of higher measurement precision become available (Aldering e.a. 2006). For  $0 \leq z \leq 1.2$  the Hubble diagrams in the two models differ only very little. After an intersection of the two curves close to  $z \approx 1.2$ , they have a noticable and increasing difference. (The standard model predicts a more rapid decrease of luminosities.) With sufficiently far and/or precise measurements of SNI beyond  $z \approx 1.2$ , we expect a statistically significant discrimination criterion between the two model classes.

The Einstein-Weyl geometry with  $\zeta \approx 2.6$  indicates relative energy densities  $\Omega_m \approx 2.4$  and  $\Omega_\Lambda \approx 1.2$ . Before we draw rash conclusions from the unexpected high mass density values, we have to realize that the upper limit of baryonic mass density of the standard approach is no longer valid in

equilibrium cosmology (from this vantage point, primordial nucleosynthesis appears as a theoretical artefact of the standard approach). A much higher amount of *classical* matter, molecular hydrogen and low energy hydrogen plasma, both distributed over large inter cluster regions, astronomically extremely difficult to trace and unobservable by the dynamical method of cluster dynamics, may very well explain such a high value. The homogeneity of the mass distribution could even be much higher than estimated at present, because clusters and super clusters would lose their predominance as tracers for dark matter (Peebles 2004).

With  $\zeta \approx 2.6$ , the energy of the vacuum boson can be determined up to the unknown coupling constant  $\gamma$  or up to

$$\tilde{\gamma} := \frac{8\pi}{c\hbar}\gamma$$

by equ. (37),

$$(59) \quad (m_0 c^2)^2 \tilde{\gamma} = E_{Pl}^2 H_1^2 (\zeta + 1) \approx 3 \text{ eV}^2 \text{ cm}^{-2}.$$

If vacuum gauge still holds for high energy experiments, the presently expected order of magnitude for the Higgs boson,  $m_H \sim 10^{11} \text{ eV}$ , would imply

$$\tilde{\gamma} \sim 10^{-22} \text{ cm}^{-2}, \quad \gamma \sim 10^{-27} \text{ eV cm}^{-1}.$$

But it is unclear, whether vacuum gauge remains applicable under these conditions.<sup>17</sup>

Some of the other empirical data prefer Weyl universes more clearly (quasar magnitudes, quasar frequencies, Pioneer anomaly),<sup>18</sup> while still other observational evidence speaks in favour of the expanding space cosmology, or may speak in favour of it in the near future (most importantly, but still undecidedly star formation rate). Contrary to a general creed, neither the exact Planck characteristic of the microwave background nor its anisotropy structure can serve as a differentiating criterion between the expanding and the equilibrium paradigms. They go well in hand with both. A clear discrimination between the different approaches may rather be expected from the antenna systems, designed for tracing the conjectured 21 cm line of non-ionized monatomic hydrogen at redshifts above  $z \approx 7$ . If the latter will be observed within the next 5 to 10 years with intensity significantly above average, the standard conjecture of a reionization period for cosmic hydrogen, somewhere in the interval  $7 < z < 30$ , will be empirically confirmed. That would be a decisive empirical triumph over most (all?) competing approaches. *If the experiment turns out null, the outcome will be converse.*

In our context of equilibrium cosmologies of case 2 fluids, the present curvature values (58) determined by supernovae data indicate the following value for the hypothetical binding energy exponential (57):

$$(60) \quad \alpha \in [0.205, 0.218] \quad \text{or} \quad \alpha \approx 0.211 \pm 0.07$$

We have to leave it open, whether we can assign any sense beyond the methodological (“toy”) exploration to these model values.

<sup>17</sup>If so, it could indicate an additional coupling of  $\phi$  to quantum matter fields or of a non-constant  $G_N$ .

<sup>18</sup>Cf. (Scholz 2007, Scholz 2005); an updated discussion of present data is in preparation.

In any case, Weyl universes, and in particular case 2 equilibrium cosmologies, lead to a striking correspondence with empirical knowledge of different origin and on quite different levels (supernovae, quasar magnitudes, quasar frequencies, and the unexpected Pioneer frequency shift). Even if the case 2 Ansatz for self binding energy may not yet be a finally realistic one and of predominantly methodological value, we have to keep in mind that the structural (geometrical) result of Weyl universes seem to be typical for *any* self binding Ansatz which leads to equilibrium geometries (cf. thm. 1). The empirical properties of Weyl universes, which have yet only been partially explored, are of much broader import than the specific assumptions of case 2. This is a sufficient reason for further investigating Weyl geometric equilibrium geometries from various viewpoints.

Moreover, we should not exclude that scale covariant gravity may become a clue for a better understanding of the coupling of matter and interaction fields to gravity. Drechsler's Higgs mechanism might be the first sign from the tip of an iceberg, which is waiting to be discovered.

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